# Damping of zero sound in Luttinger liquids

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**Abstract.** We calculate the damping $\gamma_q$ of collective density oscillations (zero sound) in a one-dimensional Fermi gas with dimensionless forward scattering interaction $F$ and quadratic energy dispersion $k^2/2m$ at zero temperature. Using standard many-body perturbation theory, we obtain $\gamma_q$ from the expansion of the inverse irreducible polarization to first order in the effective screened (RPA) interaction. For wave-vectors $|q|/k_F \ll F$ (where $k_F = mv_F$ is the Fermi wave-vector) we find to leading order $\gamma_q \propto |q|^3/(v_F m^2)$. On the other hand, for $F \ll |q|/k_F$ most of the spectral weight is carried by the particle-hole continuum, which is distributed over a frequency interval of the order of $q^2/m$. We also show that zero sound damping leads to a finite maximum proportional to $|k - k_F|^{-2+2\eta}$ of the charge peak in the single-particle spectral function, where $\eta$ is the anomalous dimension. Our prediction agrees with photoemission data for the blue bronze $K_{0.3}MoO_3$. We comment on other recent calculations of $\gamma_q$.

**PACS.** 71.10.Pm Fermions in reduced dimensions (anyons, composite fermions, Luttinger liquid, etc.) – 71.10.-w Theories and models of many-electron systems

## 1 Introduction

The normal metallic state of interacting electrons in one spatial dimension has rather exotic properties, which are summarized under the name Luttinger liquid behavior: the absence of a discontinuity in the momentum distribution function at the Fermi surface, a vanishing density of states at the Fermi energy, and an unusual line-shape of the single-particle spectral function, characterized by algebraic singularities and separate peaks for spin- and charge excitations [1,2]. To discover these features und study them quantitatively, it has been extremely useful to have an exactly solvable effective low energy model for interacting one-dimensional clean metals, the so-called Tomonaga-Luttinger model (TLM) [3,4]. The exact solubility of the TLM relies on two crucial assumptions: the linearization of the energy dispersion for momenta close to the two Fermi momenta $\pm k_F$, and the restriction to two-body scattering processes involving only momentum transfers small compared with $k_F$ (forward scattering). The single-particle Green function is then most conveniently obtained via bosonization [1,2]. Alternatively, the Ward identity associated with the separate number conservation at each Fermi point can be used to derive a closed equation for the single-particle Green function [5]. Within the Ward-identity approach, it is also straightforward to show that collective density oscillations can propagate without damping in the TLM, so that the random-phase approximation

(RPA) for the density-density correlation function $\Pi(q, \omega)$ is exact. As a consequence the dynamic structure factor $S(q, \omega) = \pi^{-1} \mathrm{Im} \Pi(q, \omega + i0)$ of the TLM exhibits a sharp $\delta$-function peak. Focusing for simplicity on the spinless TLM with forward scattering interactions $g_2 = g_4 = f_0$ in "g-ology"-notation [1,2], the dynamic structure factor is

$$S_{\mathrm{TLM}}(q, \omega) = Z_q \delta(\omega - \omega_q), \qquad (1)$$

where $Z_q = |q|/(2\pi\sqrt{1+F})$ and $\omega_q = v_c|q|$. Here $F = f_0/(\pi v_F)$ is the relevant dimensionless interaction, $v_c = v_F\sqrt{1+F}$ is the velocity of the collective charge excitations (zero sound, abbreviated by ZS from now on), and $v_F$ is the Fermi velocity.

The simple result (1) is a consequence of the approximations inherent in the definition of the TLM: the linearization of the energy dispersion and the restriction to forward scattering interactions. In more realistic models, we expect that the ZS mode acquires a finite width. How does the line-shape of $S(q, \omega)$ change if we do not linearize the energy dispersion? Since the RPA is exact for linear energy dispersion, it is reasonable to use the RPA as a starting point for quadratic dispersion and to try to calculate the corrections to the RPA perturbatively. At the first sight it seems that this problem can be solved within the usual bosonization approach by treating the effective boson-interactions due to the band curvature within conventional many-body perturbation theory for the bosonized problem [1,6,7]. However, for frequencies close to $\omega_q$ this seems not to be possible, and infinite

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re-summations are necessary [8,9]. In fact, there are conflicting results for the $q$-dependence of the damping $\gamma_q$ of the ZS mode of one-dimensional fermions in the literature: while Capurro et al. [10] found $\gamma_q \propto q^3$, Samokhin [11], Pustilnik and co-authors [12–14] and Pereira et al. [15] obtained $\gamma_q \propto q^2$. On the other hand, Teber [16] recently showed that the attenuation rate of an acoustic mode embedded in the two-pair continuum of the dynamic structure factor scales as $q^3$, which is consistent with $\gamma_q \propto q^3$.

Let us briefly discuss the recent works by Pustilnik et al. [13,14] and by Pereira et al. [15], both of which found that the damping scales as $q^2$. In reference [15] it has been shown that the width $\gamma_q$ of the peak of the longitudinal structure factor of the XXZ spin-chain is proportional to $q^2$. This is not necessarily in conflict with $\gamma_q \propto q^3$ for the TLM with band curvature, because the XXZ-chain is equivalent to a system of spinless fermions on a lattice. In contrast to the forward scattering processes of the TLM, the interaction in this model has also scattering processes involving momentum transfers of the order of $k_F$. Although in the Luttinger liquid regime of the XXZ-chain these processes are irrelevant in the renormalization group sense, their effect on non-universal quantities like $\gamma_q$ might be essential. A similar argument applies also to reference [14], where the dynamic structure factor of the Calogero-Sutherland model is shown to differ from zero only in a finite interval of frequencies of the width proportional to $q^2/m$. The Fourier transform $f_q$ of the interaction in the Calogero-Sutherland model is proportional to $|q|/m$ for all momentum transfers $q$, so that scattering is suppressed for small $q$. On the other hand, the RPA and our strategy of calculating perturbative corrections to the RPA can only be expected to be accurate if the interaction $f_q$ is strongest for small $q$. We therefore believe that the dynamic structure factor of the Calogero-Sutherland model [14] does not represent the generic behavior of the dynamic structure factor of one-dimensional Fermi systems with dominant forward scattering, as given by the TLM with curvature.

As far as the calculation in reference [13] is concerned, it is based on an infinite re-summation of the apparent leading singularities in the weak coupling expansion, using an analogy with the X-ray problem. We believe that this procedure does not properly take into account the asymptotic Ward-identity which guarantees the cancellation of all singularities in the limit $1/m \to 0$. Moreover, the renormalization of the real part of the energy of the ZS mode, which might be essential to obtain the correct damping [17,18], has not been properly taken into account in reference [13].

Within the framework of diagrammatic many-body perturbation theory, the standard approach [19] to calculate corrections to the RPA is based on the evaluation of the three Feynman diagrams shown in Figure 1, which represent the leading corrections to the irreducible polarization in an expansion in powers of the RPA interaction. Surprisingly, these diagrams have never been evaluated for the TLM with band curvature, which we shall do in this work. It seems that, at least for small interactions, this
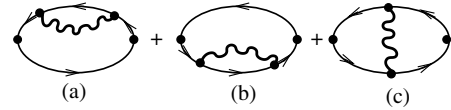


**Fig. 1.** Leading interaction corrections to the irreducible polarization in an expansion in powers of the RPA interaction. Solid arrows are non-interacting Green functions and wavy lines denote the RPA interaction.

approach should be sufficient to estimate the damping of the ZS mode. We shall further comment on the accuracy of this approach in Section 4.

## 2 RPA in one dimension with quadratic dispersion

It is instructive to consider first the density-density correlation function for fermions with quadratic energy dispersion $\epsilon_k = k^2/2m$ within the RPA, where

$$\Pi_{\mathrm{RPA}}^{-1}(Q) = f_0 + \Pi_0^{-1}(Q). \tag{2}$$

For convenience we use the Matsubara formalism and collective labels $Q = (i\omega, q)$ for wave-vector $q$ and bosonic Matsubara frequency $i\omega$. To make contact with the usual distinction between left-moving and right-moving fermions in the TLM, we write the non-interacting polarization as $\Pi_0(Q) = \sum_{\alpha=\pm} \Pi_0^{\alpha}(Q)$, where $\alpha = +$ refers to right-moving fermions (with velocity $v_k = k/m > 0$) while $\alpha = -$ denotes left-moving fermions ($v_k < 0$). At finite temperature $T$ for a system of length $L$,

$$\Pi_0^{\alpha}(Q) = -\frac{T}{L} \sum_K \Theta^{\alpha}(k) G_0^{\alpha}(K) G_0^{\alpha}(K+Q), \tag{3}$$

where $G_0^{\alpha}(K) = [i\tilde{\omega} - \xi_k^{\alpha}]^{-1}$ is the free Matsubara Green function and $\xi_k^{\alpha} = \epsilon_{\alpha k_F + k} - \epsilon_{k_F} = \alpha v_F k + k^2/2m$ is the free excitation energy. The label $K = (i\tilde{\omega}, k)$ consists of fermionic Matsubara frequency $i\tilde{\omega}$ and momentum $k$, which is measured relative to $\alpha k_F$. The cutoff function $\Theta^+(k)$ restricts the range of the $k$-integration to the interval $[-k_F, \infty)$ associated with right-moving fermions, while $\Theta^-(k)$ selects the interval $(-\infty, k_F]$ corresponding to left-moving fermions. All degrees of freedom are taken into account in this way. At $T = 0$ the integrations in equation (3) are easily carried out,

$$\Pi_0^{\alpha}(Q) = \alpha \frac{m}{2\pi q} \ln \left[ \frac{\alpha v_F q - i\omega + q^2/2m}{\alpha v_F q - i\omega - q^2/2m} \right], \tag{4}$$

so that dynamic structure factor $S_{\mathrm{RPA}}(q, \omega) = \pi^{-1} \mathrm{Im} \Pi_{\mathrm{RPA}}(q, \omega + i0)$ can be calculated analytically. For any finite value of the interaction $S_{\mathrm{RPA}}(q, \omega)$ consists of two contributions: a continuum part due to partice-hole excitations, and a sharp $\delta$-peak $S_{\mathrm{RPA}}^{\mathrm{zs}}(q, \omega) = Z_q \delta(\omega - \omega_q)$ corresponding to the undamped collective ZS mode, see Figure 2. The weight $Z_q$ and the dispersion $\omega_q$ of the ZS
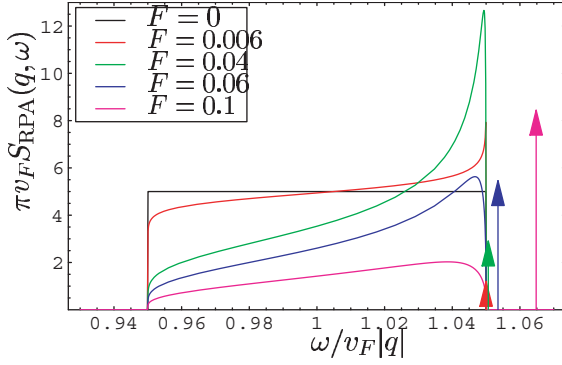
**Fig. 2.** RPA dynamic structure factor for quadratic energy dispersion, $|q|/k_F = 0.1$, and different values of the dimensionless interaction $F$. The arrows denote the location of the $\delta$-peak associated with the ZS mode, the length of the arrows being proportional to the relative weight $W_q$ of the ZS peak in the $f$-sum rule, see equation (5).

mode are within RPA

$$Z_q = \frac{v_F q^2}{2\pi\omega_q} W_q, \quad W_q = (\tilde{q}/F)^2 / \sinh^2(\tilde{q}/F), \quad (5)$$

$$\omega_q = v_F|q|[1 + \tilde{q}\coth(\tilde{q}/F) + (\tilde{q}/2F)^2]^{1/2}, \quad (6)$$

where $\tilde{q} = q/k_F$. Obviously, the limits of vanishing band curvature ($|\tilde{q}| = |q|/mv_F \to 0$) and vanishing interaction ($F \to 0$) do not commute: for $|\tilde{q}| \gg F$ the weight $Z_q$ of the ZS mode is exponentially small, so that the particle-hole part of the dynamic structure is dominant. For $F \to 0$ the latter approaches a box-function centered at $\omega = v_F|q|$ of width $q^2/m$ and height $m/(2\pi|q|)$, see Figure 2. In the opposite limit $|\tilde{q}| \ll F$ the ZS peak is dominant. To quantify this, note that the dimensionless factor $W_q$ defined in equation (5) can be identified with the relative weight of the ZS peak in the $f$-sum rule $\int_0^\infty d\omega\, \omega\, S(q,\omega) = v_F q^2/2\pi$. From Figure 3 it is clear that for $|\tilde{q}| \ll F$ the contribution of the particle-hole continuum to the $f$-sum rule is indeed negligible (actually, it is of order $(\tilde{q}/F)^2 \ll 1$), and that the crossover between the particle-hole regime $|\tilde{q}| \gg F$ and the ZS regime $|\tilde{q}| \ll F$ occurs at $|\tilde{q}| \approx F$.

The line-shape of the particle-hole continuum and its position relative to the ZS mode are probably not correct within the RPA. In fact, recently Schönhammer [17] has shown that if one uses in the RPA bubbles Hartree-Fock propagators instead of bare ones, then the energy of the ZS mode is *smaller* than the energy of the particle-hole continuum. Moreover, multi-pair particle-hole excitations neglected within the RPA will wash out the sharp thresholds predicted by the RPA and generate some small spectral weight for all frequencies [20]. However, for sufficiently small $q$ almost the entire spectral weight is carried by the ZS mode, so that in this limit we may neglect the particle-hole continuum.

## 3 Leading correction to the RPA

Because for $|q|/k_F \ll F$ the spectrum of the density fluctuations is dominated by the collective ZS mode, we
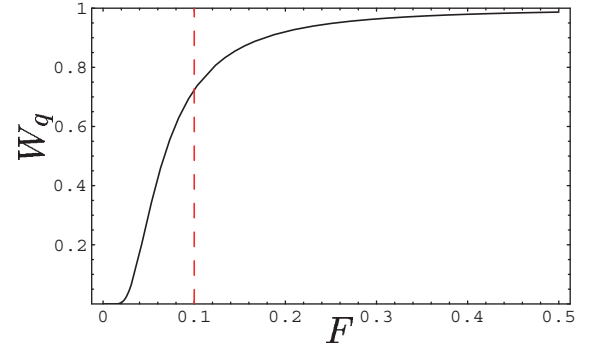


**Fig. 3.** Weight $W_q$ of the ZS mode in the $f$-sum rule in RPA for $q/k_F = 0.1$ (dashed line) as a function of $F$, see equation (5).

expect that the exact structure factor for frequencies $\omega$ close to the exact ZS frequency $\omega_q$ can be approximated by a Lorentzian,

$$S(q,\omega) \approx \pi^{-1} Z_q \gamma_q [(\omega - \omega_q)^2 + \gamma_q^2]^{-1}. \quad (7)$$

In terms of the irreducible polarization $\Pi_*(Q)$ defined via $\Pi^{-1}(Q) = f_0 + \Pi_*^{-1}(Q)$ the inverse weight of the ZS peak is given by

$$Z_q^{-1} = f_0^2 \mathrm{Re}\Pi_*'(q,\omega_q), \quad (8)$$

and the damping is

$$\gamma_q = \mathrm{Im}\Pi_*(q,\omega_q + i0)/\mathrm{Re}\Pi_*'(q,\omega_q), \quad (9)$$

where $\Pi_*'(q,\omega) = \partial\Pi_*(q,\omega)/\partial\omega$.

Since $\gamma_q = 0$ within RPA, we need to go beyond RPA to estimate $\gamma_q$. This is usually done [19] by expanding the $\Pi_*(Q)$ in powers of the RPA interaction $f_{\mathrm{RPA}}(Q) = f_0[1 + f_0\Pi_0(Q)]^{-1}$. Before presenting an explicit calculation, let us give a simple argument for the expected $q$-dependence of $\gamma_q$. Within the functional bosonization approach [7], $\Pi^\alpha(Q)$ can be written as a single-particle Green function of a real bosonic quantum field $\rho_Q^\alpha$ representing the density fluctuations,

$$\Pi^\alpha(Q) \propto \int \mathcal{D}[\rho^\alpha] e^{-S_{\mathrm{eff}}[\rho^\alpha]} \rho_{-Q}^\alpha \rho_Q^\alpha. \quad (10)$$

For linear dispersion the effective action $S_{\mathrm{eff}}[\rho^\alpha]$ is quadratic, so that equation (10) yields the RPA result, which is exact for the TLM. Non-linear terms in the energy dispersion renormalize the quadratic part of $S_{\mathrm{eff}}[\rho^\alpha]$ and give rise to cubic, quartic, and higher order retarded interaction vertices [7], which all generate corrections to the RPA. We parameterize these corrections in terms of an irreducible self-energy $\Sigma^\alpha(Q)$. Assuming that $\Sigma^\alpha(Q)$ is analytic for small $Q$ (this assumption relies on the cancellation of all singularities in the symmetrized closed fermion loops [18]), and taking into account that $\Sigma^\alpha(Q) = \Sigma^\alpha(-Q)$ (which follows from the invariance of the right-hand side of Eq. (10) under $Q \to -Q$), we conclude that $\Sigma^\alpha(q,\omega_q + i0) = c_0 + c_2(q/k_F)^2 + O(q^4)$, with real $c_0$ and complex $c_2$. But the relation between the density fields $\rho_Q^\alpha$ and the Tomonaga-Luttinger bosons

whose damping $\gamma_q$ we are seeking involves an extra factor of $|q|^{1/2}$ (see for example p. 57 of Ref. [7]), so that we expect $\gamma_q \propto |q||\text{Im}\Sigma^\alpha(q, \omega_q + i0) \propto |q|^3$.

We now confirm this result by explicitly calculating $\Sigma^\alpha(Q)$ to first order in the RPA interaction. The diagrams contributing to $\Pi_*(Q)$ to this order are shown in Figure 1. Writing again $\Pi_*(Q) = \sum_\alpha \Pi_*^\alpha(Q)$ we have $\Pi_*^\alpha(Q) \approx \Pi_0^\alpha(Q) + \Pi_1^\alpha(Q)$, with $\Pi_1^\alpha(Q) = \Pi_{1s}^\alpha(Q) + \Pi_{1v}^\alpha(Q)$. The sum of the self-energy corrections shown in Figures 1a, 1b is

$$\Pi_{1s}^\alpha(Q) = \frac{T^2}{L^2} \sum_{K,Q'} \Theta^\alpha(k) f_{\text{RPA}}(Q')[G_0^\alpha(K)]^2$$
$$\times G_0^\alpha(K+Q')[G_0^\alpha(K+Q) + G_0^\alpha(K-Q)], \quad (11)$$

and the vertex correction in Figure 1c is

$$\Pi_{1v}^\alpha(Q) = \frac{T^2}{L^2} \sum_{K,Q'} \Theta^\alpha(k) f_{\text{RPA}}(Q') G_0^\alpha(K) G_0^\alpha(K+Q)$$
$$\times G_0^\alpha(K+Q') G_0^\alpha(K+Q+Q'). \quad (12)$$

To regularize some of the $q'$-integrations, we introduce a momentum transfer cutoff $q_c \ll k_F$ which restricts the integration range to $|q'| \leq q_c$. The imaginary part of these expressions is not sensitive to $q_c$. To evaluate equations (11, 12) in the limit $T \to 0$ it is convenient to write [7]

$$f_{\text{RPA}}(q, i\omega) = f_0 - f_0^2 \int_0^\infty d\omega' \frac{2\omega' S_{\text{RPA}}(q, \omega')}{\omega'^2 + \omega^2}. \quad (13)$$

The frequency integrations in equations (11) and (12) can then be carried out exactly. The result is

$$\Pi_{1s}^\alpha(q, i\omega) = \frac{f_0}{L^2} \sum_{k,q'} \Theta^\alpha(k) \theta(-\xi_{k+q'}^\alpha)$$
$$\times \left[ \frac{\delta(\xi_k^\alpha)}{\xi_{k+q}^\alpha - i\omega} + \frac{\text{sgn}(\xi_k^\alpha)\theta(-\xi_k^\alpha \xi_{k+q}^\alpha)}{[\xi_k^\alpha - \xi_{k+q}^\alpha + i\omega]^2} \right]$$
$$- \frac{f_0^2}{L^2} \sum_{k,q'} \Theta^\alpha(k) \int_0^\infty d\omega' \left\{ \frac{S_{\text{RPA}}(q', \omega')}{\xi_k^\alpha - \xi_{k+q}^\alpha + i\omega} \right.$$
$$\times \left[ \frac{\text{sgn}(\xi_{k+q'}^\alpha)\delta(\xi_k^\alpha)}{|\xi_{k+q'}^\alpha| + \omega'} + \frac{\text{sgn}(\xi_k^\alpha)\theta(-\xi_k^\alpha \xi_{k+q'}^\alpha)}{[|\xi_k^\alpha| + |\xi_{k+q'}^\alpha| + \omega']^2} \right]$$
$$+ \frac{S_{\text{RPA}}(q', \omega')}{[\xi_k^\alpha - \xi_{k+q}^\alpha + i\omega]^2} \left[ \frac{\theta(-\xi_k^\alpha \xi_{k+q'}^\alpha)}{|\xi_k^\alpha| + |\xi_{k+q'}^\alpha| + \omega'} \right.$$
$$\left. \left. - \frac{\theta(-\xi_{k+q}^\alpha \xi_{k+q'}^\alpha)}{|\xi_{k+q}^\alpha| + |\xi_{k+q'}^\alpha| + \omega' + i\omega\, \text{sgn}(\xi_{k+q'}^\alpha)} \right] \right\}$$
$$+ [(q, i\omega) \to (-q, -i\omega)], \quad (14)$$

$$\Pi_{1v}^\alpha(q, i\omega) = -\frac{f_0}{L^2} \sum_{k,q'} \Theta^\alpha(k)$$
$$\times \frac{\theta(-\xi_{k+q'}^\alpha)\text{sgn}(\xi_k^\alpha)\theta(-\xi_k^\alpha \xi_{k+q}^\alpha)}{[\xi_{k+q'}^\alpha - \xi_{k+q+q'}^\alpha + i\omega][\xi_k^\alpha - \xi_{k+q}^\alpha + i\omega]}$$
$$+ \frac{f_0^2}{L^2} \sum_{k,q'} \Theta^\alpha(k) \int_0^\infty d\omega'$$
$$\times \frac{S_{\text{RPA}}(q', \omega')}{[\xi_k^\alpha - \xi_{k+q}^\alpha + i\omega][\xi_{k+q'}^\alpha - \xi_{k+q'+q}^\alpha + i\omega]}$$
$$\times \left[ \frac{\theta(-\xi_k^\alpha \xi_{k+q'}^\alpha)}{|\xi_k^\alpha| + |\xi_{k+q'}^\alpha| + \omega'} \right.$$
$$\left. - \frac{\theta(-\xi_{k+q}^\alpha \xi_{k+q'}^\alpha)}{|\xi_{k+q}^\alpha| + |\xi_{k+q'}^\alpha| + \omega' + i\omega\, \text{sgn}(\xi_{k+q'}^\alpha)} \right]$$
$$+ [(q, i\omega) \to (-q, -i\omega)]. \quad (15)$$

For linearized dispersion, $\xi_k^\alpha \approx \alpha v_F k$, we have verified that $\Pi_{1s}^\alpha(q, i\omega) + \Pi_{1v}^\alpha(q, i\omega) = 0$, in agreement with the closed loop theorem [5,7]. With quadratic dispersion $\xi_k^\alpha = \alpha v_F k + k^2/2m$ this cancellation is not perfect. Assuming that the band curvature $1/m$ is small, we may approximate $S_{\text{RPA}}(q, \omega) \approx S_{\text{RPA}}(q, \omega)|_{1/m=0} = S_{\text{TLM}}(q, \omega)$ in equations (14) and (15), because the $m$-dependence of $S_{\text{RPA}}(q, \omega)$ is irrelevant for the cancellation between self-energy and vertex corrections in the limit $1/m \to 0$. One should keep in mind, however, that this is only a good approximation as long as the dynamic structure factor is dominated by the ZS mode, which is the case for $|q|/k_F \ll F$. We shall from now on focus on this regime. With $S_{\text{TLM}}(q, \omega)$ given in equation (1), the frequency integration is trivial. However, the resulting expression is ill-defined, because $\text{Im}\Pi_1^\alpha(q, \omega + i0)$ contains singular terms proportional to $\delta(\omega - \xi_q^\alpha)$. The fact that the direct expansion of $\Pi_*(q, \omega)$ in powers of the interaction generates unphysical singularities has been noticed long time ago [19]. These singularities can be avoided by expanding the *inverse* polarization $[\Pi_*^\alpha(Q)]^{-1}$ in powers of the interaction, $[\Pi_*^\alpha(Q)]^{-1} = [\Pi_0^\alpha(Q)]^{-1} - \Sigma_1^\alpha(Q)$, where to first order in the RPA interaction $\Sigma_1^\alpha(Q) = [\Pi_0^\alpha(Q)]^{-2}\Pi_1^\alpha(Q)$, with $\Pi_1^\alpha(Q) = \Pi_{1s}^\alpha(Q) + \Pi_{1v}^\alpha(Q)$ given in equations (11) and (12). The point is that according to equation (10) $\Pi^\alpha(Q)$ can be viewed as a bosonic single-particle Green function, which should never be calculated directly, because its expansion in powers of the interaction usually contains unphysical singularities associated with the free Green function. On the other hand, the expansion of the *inverse* Green function (i.e., the self-energy) is expected to be regular. Ignoring the renormalization of the real part of the ZS dispersion, we approximate

$$\gamma_q \approx \text{Im}\Sigma_1^{\alpha_q}(q, \omega_q + i0)[\Pi_0^{\alpha_q}(q, \omega_q)]^2 / \Pi_0'(q, \omega_q), \quad (16)$$

where $\alpha_q = \text{sgn}(q)$. For simplicity we consider only the leading behavior of $\gamma_q$ in an expansion in powers of $q$ and $1/m$. The integrations over the momenta $k$ and $q'$ in equations (14) and (15) can then be carried out exactly. For
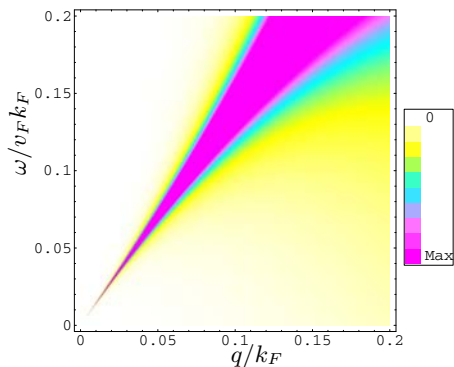
**Fig. 4.** Dynamic structure factor in Lorentzian approximation for $F = 1$, see equations (5–7) and (17).

$|q|/k_F \ll \min\{F, 1\}$ we obtain the expected result

$$\gamma_q \approx \frac{\pi}{8} \frac{F^3}{\sqrt{1+F}[1+\sqrt{1+F}]^4} \frac{|q|^3}{v_F m^2}. \qquad (17)$$

An intensity plot of the corresponding structure factor (7) in Lorentzian approximation is shown in Figure 4.

Our result agrees qualitatively with reference [10], which found $\gamma_q \propto F|q|^3/v_F m^2$, but we disagree with references [11,12], who obtained $\gamma_q \propto q^2/m$. Possibly the discrepancy with equation (17) is related to the non-commutativity of the limits $q \to 0$ and $F \to 0$, which is obvious from equations (5) and (6). Our result $\gamma_q \propto q^3$ is valid for $|q|/k_F \ll F$, which for fixed $|q|$ requires a minimum strength $F$ of the interaction, but for fixed $F > 0$ is always satisfied for sufficiently small $q$. In the opposite limit $F \ll |q|/k_F$, where most of the spectral weight is carried by single particle-hole excitations, our approach also yields $\gamma_q \propto q^2/m$. The $q^3$-scaling of $\gamma_q$ is consistent with a recent result by Teber [16], who showed that the attenuation rate of a coherent acoustic mode embedded in the two-pair excitation continuum scales as $q^3$.

$S(q, \omega)$ can be measured using X-ray scattering. Unfortunately, for small $q$ the intensity is very low, so that an experimental verification of Figure 4 seems to be difficult. However, the ZS damping has a dramatic effect on the shape of the single-particle spectral function $A(k_F + q, \omega)$ for $\omega \approx \omega_q$, which can be measured via photoemission. For the TLM (including now the spin degree of freedom) one finds for $\omega \to \omega_q = v_c|q|$ an algebraic singularity [21], $A(k_F + q, \omega) \propto |q(\omega - \omega_q)|^{(\eta-1)/2}$, which is a consequence of the undamped ZS propagation. Here $\eta$ is the anomalous dimension. The damping of the ZS mode washes out this singularity. The sensitivity of the spectral line-shape for $\omega \approx \omega_q$ to non-universal perturbations neglected in the TLM has previously been noticed by Meden [22]. In fact, from the expression for the single-particle Green function derived via functional bosonization [6,7] we estimate that the photoemission intensity exhibits for $|\omega - \omega_q| \ll \gamma_q$ a finite [23] maximum, $I_{\max}(q) \propto |q\gamma_q|^{(\eta-1)/2} \propto |q|^{-2+2\eta}$. In Figure 5 we compare this prediction with photoemission data [24] for the "blue bronze" $K_{0.3}MoO_3$. The resulting value of $\eta$ is consistent with previous estimates [24] be-
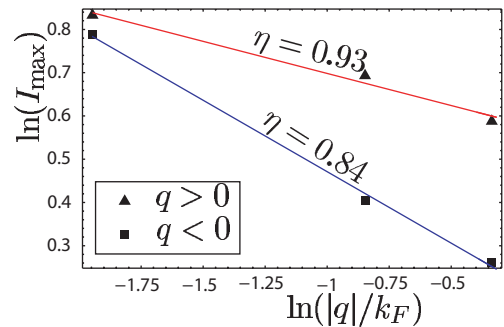


**Fig. 5.** Maximum intensity $I_{\max}(q)$ of the charge peak in the photoemission spectrum of the quasi one-dimensional metal $K_{0.3}MoO_3$. The points are data from Figure 8 of reference [24], assuming that $q = 0$ corresponds to an emission angle $\theta = 7°$. The lines are fits to $I_{\max}(q) \propto |q|^{-2+2\eta}$. Our theory is valid only for $|q| \ll k_F$, which might explain the discrepancy between positive and negative $q$ for larger $|q|$.

tween 0.7 and 0.9 for this material. However, the data [24] were taken at $T = 300$ K, so that deviations from our $T = 0$ theory are expected [23]. Refined photoemission data probing the $T = 0$ regime of a Luttinger liquid would be useful.

## 4 Summary and conclusions

In summary, using a standard diagrammatic approach we have calculated the damping $\gamma_q$ of the collective charge mode (zero sound) in a Luttinger liquid due to the non-linearity in the energy dispersion. Our result $\gamma_q \propto |q|^3/v_F m^2$ suggests that ZS is indeed a well defined elementary excitation in a Luttinger liquid. We have pointed out that the photoemission line-shape close to the charge peak is very sensitive to the ZS damping and have made a prediction for the height of the charge peak which agrees with experiment [24]. ZS damping is also crucial to understand Coulomb drag experiments in quantum wires [12].

We emphasize that our result is based on the expansion of the inverse irreducible polarization to first order in the RPA interaction, using standard many-body perturbation theory. For Fermi systems in three dimensions this approach has been quite successful [19], and we have shown here that also in one dimension the perturbative calculation of the ZS damping does not give rise to any singularities, at least to first order in an expansion in powers of the RPA interaction. However, according to reference [13] higher order corrections in the *bare* interaction (not contained in our expansion to first order in the RPA interaction) generate new singularities which, when properly re-summed, lead to a much larger damping, $\gamma_q \propto q^2/m$. We suspect that the re-summation procedure proposed in reference [13] is unreliable, because it does not properly take into account the closed loop theorem [5,7]. We are currently investigating this problem using a functional renormalization group approach [18,25] with momentum transfer cutoff, where corrections to the RPA can be calculated systematically even in the presence of infrared singularities.

Finally, let us emphasize that, in contrast to expansions using conventional bosonization [8,9,11], our approach is not based on a direct expansion of the dynamic structure factor in powers of $1/m$. This is obvious from the fact that in the non-interacting limit we recover the exact dynamic structure factor of the free Fermi gas with quadratic energy dispersion. Our approach can be formally justified with the help of functional bosonization [7,18], where the vertices of the effective bosonized theory are given by symmetrized closed fermion loops and the effective small parameter which controls a perturbative expansion is proportional to the combination $F q_c/k_F$, where the range $q_c$ of the interaction in momentum space must be small compared with $k_F = m v_F$.

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